Using Qualitative Reasoning in Modelling Consensus in Group Decision-Making

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Abstract

Ordinal scales are commonly used in rating and evaluation processes. These processes usually involve group decision making by means of an experts' committee. In this paper a mathematical framework based on the qualitative model of the absolute orders of magnitude is considered. The entropy of a qualitatively described system is defined in this framework. On the one hand, this enables us to measure the amount of information provided by each evaluator and, on the other hand, the coherence of the evaluation committee. The new approach is capable of managing situations where the assessment given by experts involves different levels of precision. The use of the proposed measures within an automatic system for group decision making will contribute towards avoiding the potential subjectivity caused by conflicts of interests of the evaluators in the group.

Introduction

Nowadays, accreditation, audit, or rating agencies are dealing with a huge problem. Most committees are unable to ensure their legitimacy. Recent events have questioned the integrity of the rating agencies and their processes, and scandal stories about them have appeared in press and media.

This work is intended to be a first step towards the definition of evaluation measures in the group decision processes. To this end we introduce an approach based on qualitative reasoning models and the concept of entropy in order to measure the degree of coherence reached by an evaluation group.

Qualitative Reasoning (QR) is a sub-area of Artificial Intelligence that seeks to understand and explain human beings' ability for qualitative reasoning (Forbus 1996), (Kuipers 2004). The main objective is to develop systems that permit operating in conditions of insufficient numerical data or in the absence of such data. As indicated in (Travé-Massuyès and Dague 2003), this could be due to both a lack of information as well as to an information overload. A main goal of Qualitative Reasoning is to tackle problems in such a way that the principle of relevance is preserved; that is to say, each variable has to be valued with the level of precision required (Forbus 1984). It is not unusual for a situation to arise in which it is necessary to work simultaneously with different levels of precision, depending on the available information. To this end, the mathematical structures of Orders of Magnitude Qualitative Spaces (OM) were introduced.

The concept of entropy has its origins in the nineteenth century, particularly in thermodynamics and statistics. This theory has been developed from two aspects: the macroscopic, as introduced by Carnot, Clausius, Gibbs, Planck and Caratheodory; and the microscopic, developed by Maxwell and Boltzmann (Rokhlin 1967). The statistical concept of Shannon's entropy, related to the microscopic aspect, is a measure of the amount of information (Shannon 1948), (Cover and Thomas 1991).

Starting from the adaptation of the basic principles of Measure Theory (Halmos 1974), (Folland 1999) to the structure of OM (Roselló et al. 2008), this paper defines the concept of entropy within the QR framework.

Taking into account that entropy can be used to measure the amount of information, this work presents a way of measuring the amount of information given by an evaluator when describing a system by means of orders of magnitude. On the other hand, the defined entropy is applied to analyse the coherence degree of an evaluation committee in group decision making.

Section 2 presents the theoretical framework. In Section 3, the qualitative description induced by an evaluator is studied. Two operations for information aggregation and the concept of entropy in the absolute orders of magnitude spaces are defined in Section 4 and 5 respectively, and Section 6 introduces a coherence degree in group decision. The paper ends with several conclusions and outlines some proposals for future research.

Theoretical Framework

Order of magnitude models are an essential piece among the theoretical tools available for qualitative reasoning about physical systems ((Kalagnanam, Simon, and Iwasaki 1991), (Struss 1988). They aim at capturing order of magnitude commonsense ((Travé-Massuyès 1997)) inferences, such as used in the engineering world. Order of magnitude knowl-

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edge may be of two types: absolute or relative. The absolute order of magnitudes are represented by a partition of $\mathbb{R}$, each element of the partition standing for a basic qualitative class. A general algebraic structure, called Qualitative Algebra or Q-algebra, was defined on this framework ((Travé-Massuyès and Piera 1989)), providing a mathematical structure which unifies sign algebra and interval algebra through a continuum of qualitative structures built from the rougher to the finest partition of the real line. The most referenced order of magnitude Q-algebra partitions the real line into 7 classes, corresponding to the labels: Negative Large(NL), Negative Medium(NM), Negative Small(NS), Zero(0), Positive Small(PS), Positive Medium(PM) and Positive Large(PL). Q-algebras and their algebraic properties have been extensively studied ((Missier, Piera, and Travé 1989), (Travé-Massuyès and Dague 2003))

Order of magnitude knowledge may also be of relative type, in the sense that a quantity is qualified with respect to another quantity by means of a set of binary order-of-magnitude relations. The seminal relative orders of magnitude model was the formal system FOG ((Raiman 1986)), based on three basic relations, used to represent the intuitive concepts of "negligible with respect to" (Ne), "close to" (Vo) and "comparable to" (Co), and described by 32 intuition-based inference rules. The relative orders of magnitude models that were proposed later improved FOG not only in the necessary aspect of a rigorous formalisation, but also permitting the incorporation of quantitative information when available and the control of the inference process, in order to obtain valid results in the real world ((Mavrovouniotis and Stephanopoulos 1987), (Dague 1993a), (Dague 1993b)).

In ((Travé-Massuyès et al. 2002), (Travé-Massuyès and Dague 2003)) the conditions under which an absolute orders of magnitude and a relative orders of magnitude model are consistent is analysed and the constraints that consistency implies are determined and interpreted.

In (Roselló et al. 2008) a generalization of qualitative orders of magnitude was proposed to provide the theoretical basis on which to develop a Measure Theory in this context.

The *classical orders of magnitude qualitative spaces* (Travé-Massuyès and Dague 2003) verify the conditions of the generalized model introduced in (Roselló et al. 2008). These models are built from a set of ordered basic qualitative labels determined by a partition of the real line.

Let X be the real interval $[a_1, a_n)$, and a partition of this set given by $\{a_2, \dots, a_{n-1}\}$, with $a_1 < a_2 < \dots < a_{n-1} < a_n$. The set of basic labels is

$$\mathcal{S} = \{B_1, \dots, B_{n-1}\},$$

where, for $1 \le i \le n-1$, B_i is the real interval $[a_i, a_{i+1})$.
The set of indexes is $I = \{1, 2, \dots, n-1\}$.
For $1 \le i < j \le n-1$ the non-basic label $[B_i, B_j)$ is:

$$[B_i, B_j) = \{B_i, B_{i+1}, \dots, B_{j-1}\},$$

and it is interpreted as the real interval $[a_i, a_j)$.
For $1 \le i \le n-1$ the non-basic label $[B_i, B_\infty)$ is:

$$[B_i, B_\infty) = \{B_i, B_{i+1}, \dots, B_{n-1}\},$$

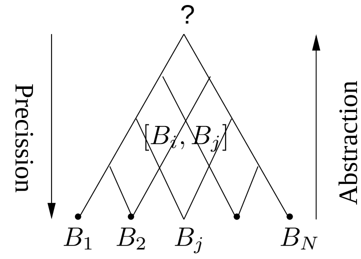

Figure 1: The space $\mathbb{S}_n$

and it is interpreted as the real interval $[a_i, a_n)$.
The complete universe of description for the Orders of Magnitude Space is the set

$$\mathbb{S}_n = \{\, [B_i, B_j) \mid B_i, B_j \in \mathcal{S}, i \le j\} \cup \{\, [B_i, B_\infty) \mid B_i \in \mathcal{S}\},$$

which is called the absolute orders of magnitude qualitative space with granularity n, also denoted $OM(n)$.

There is a partial order relation $\le_P$ in $\mathbb{S}_n$ "to be more precisely than", given by:

$$L_1 \le_P L_2 \iff L_1 \subset L_2. \qquad (1)$$

The least precise label is denoted by ? and it is the label $[B_1, B_\infty)$, which corresponds to the interval $[a_1, a_n)$.

This structure permits working with all different levels of precision from the label ? to the basic labels.

In some theoretical works, orders of magnitude qualitative spaces are constructed by partitioning the whole real line $(-\infty, +\infty)$ instead of a bounded real interval $[a_1, a_n)$. However, in most real world applications involved variables do have a lower bound a_1 and an upper bound a_n, and then values less than a_1 or greater than a_n are considered as outliers and they are not treated like any other. To introduce the classical concept of entropy by means of qualitative orders of magnitude spaces, Measure Theory is required. This theory seeks to generalize the concept of "length", "area"and "volume", understanding that these quantities need not necessarily correspond to their physical counterparts, but may in fact represent others. The main use of the measure is to define the concept of integration for orders of magnitude spaces. In (Roselló et al. 2008) measures on the generalized qualitative orders of magnitude spaces are defined.

Qualitativization induced by an evaluator

To introduce the concept of entropy by means of qualitative orders of magnitude, it is necessary to consider the qualitativization function between the set to be qualitatively described and the space of qualitative labels, $\mathbb{S}_n$.

To simplify the notation, let us express with a calligraphic letter the elements in $\mathbb{S}_n$; thus, for example, elements $[B_i, B_j)$ or $[B_i, B_\infty)$ shall be denoted as $\mathcal{E}$.
Let Λ be the set that represents a magnitude or a feature that is qualitatively described by means of the labels of $\mathbb{S}_n$. Since Λ can represent both a continuous magnitude such as position and temperature, etc., and a discrete feature such as salary and colour, etc., Λ could be considered as the range of a function

$$a : I \subset \mathbb{R} \to Y,$$

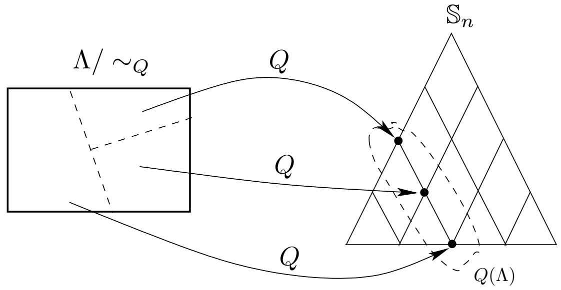

Figure 2: The qualitativization of a set Λ by means of Q.

where Y is a convenient set. For instance, if a is a room temperature during a period of time $I = [t_0, t_1]$, Λ is the range of temperatures during this period of time. Another example can be considered when $I = \{1, \ldots, n\}$ and $\Lambda = \{a(1), \ldots, a(n)\}$ are n people whose eye colour we aim to describe. In general, $\Lambda = \{a(t) = a_t \,|\, t \in I\}$.

The process of qualitativization is given by a function

$$Q : \Lambda \to \mathbb{S}_n,$$

where $a_t \mapsto Q(a_t) = \mathcal{E}_t =$ minimum label (with respect to the inclusion $\subset$) which describes a_t, i.e. the most precise qualitative label describing a_t. All the elements of the set $Q^{-1}(\mathcal{E}_t)$ are "representatives" of the label $\mathcal{E}_t$ or "are qualitatively described" by $\mathcal{E}_t$. They are qualitatively equal.

The function Q induces a partition in Λ by means of the equivalence relation:

$$a \sim_Q b \Longleftrightarrow Q(a) = Q(b).$$

This partition will be denoted by $\Lambda/\sim_Q$, and its equivalence classes are the sets $Q^{-1}(Q(a_j)) = Q^{-1}(\mathcal{E}_j)$, $\forall j \in J \subset I$. Each of these classes contains all the elements of Λ which are described by the same qualitative label.

Information aggregation

Given two qualitativizations Q and Q' of the set Λ over a space $\mathbb{S}_n$ it is natural to define two different operations between them. Intuitively speaking, one is the result of *mix* the two knowledges in a new knowledge that includes everything known about each element of Λ, and the other one is the result of taking what is *common* between the two knowledges.

The operation mix $\vee$

Definition 1 *Given two qualitativizations Q and Q', the operation $Q \vee Q'$ is a new qualitativization function $Q \vee Q' : \Lambda \to \mathbb{S}_n$ such that*

$$(Q \vee Q')(a_t) = Q(a_t) \sqcup Q'(a_t),$$

where $\sqcup$ is the connex union of labels i.e. the minimum label describing the elements of $Q^{-1}(Q(a_t))$ and the elements of $Q'^{-1}(Q'(a_t))$.

The partition

$$(\Lambda/\sim_Q) \cap (\Lambda/\sim_{Q'}) = \{X_i \cap Y_j | X_i \in \Lambda/\sim_Q, Y_j \in \Lambda/\sim_{Q'}\}.$$

is not the partition $\Lambda/\sim_{Q\vee Q'}$ because there may be $a_{t_0} \in X_{i_0} \cap Y_{j_0}$ and $a_{t_1} \in X_{i_1} \cap Y_{j_1}$ such that $(Q \vee Q')(a_{t_0}) = (Q \vee Q')(a_{t_1})$. The relation between these partitions is given by the next proposition.

Proposition 1 *Given a set Λ, the space $\mathbb{S}_n$ and two qualitativizations Q and Q', then each class of $\Lambda/\sim_{Q\vee Q'}$ is a (disjoint) union of classes of $(\Lambda/\sim_Q) \cap (\Lambda/\sim_{Q'})$:*

$$\mathrm{Class}_{Q\vee Q'}(x) = \bigcup_{y \in \mathrm{Class}_{Q\vee Q'}(x)} (\mathrm{Class}_Q(y) \cap \mathrm{Class}_{Q'}(y))$$

Proof: This set equality will be proven by double inclusion:

$\subset$) If $z \in \mathrm{Class}_{Q\vee Q'}(x)$ then it is trivial that $z \in \bigcup_{y \in \mathrm{Class}_{Q\vee Q'}(x)} (\mathrm{Class}_Q(y) \cap \mathrm{Class}_{Q'}(y))$.

$\supset$) If $z \in \bigcup_{y \in \mathrm{Class}_{Q\vee Q'}(x)} (\mathrm{Class}_Q(y) \cap \mathrm{Class}_{Q'}(y))$ then there exists $y \in \mathrm{Class}_{Q\vee Q'}(x)$ such that $Q(z) = Q(y)$ and $Q'(z) = Q'(y)$, then $(Q \vee Q')(z) = (Q \vee Q')(y) = (Q \vee Q')(x)$, whence $z \in \mathrm{Class}_{Q\vee Q'}(x)$

The last step is the proof that it is a disjoint union: let be $y, z \in \mathrm{Class}_{Q\vee Q'}(x)$, then $\mathrm{Class}_Q(y) \cap \mathrm{Class}_{Q'}(y) \cap \mathrm{Class}_Q(z) \cap \mathrm{Class}_{Q'}(z) = \emptyset$ or $\mathrm{Class}_Q(y) \cap \mathrm{Class}_{Q'}(y) = \mathrm{Class}_Q(z) \cap \mathrm{Class}_{Q'}(z)$. In effect:

$$t \in \mathrm{Class}_Q(y) \cap \mathrm{Class}_{Q'}(y) \cap \mathrm{Class}_Q(z) \cap \mathrm{Class}_{Q'}(z) \Rightarrow$$
$$\Rightarrow Q(t) = Q(y), Q'(t) = Q'(y), Q(t) = Q(z), Q'(t) = Q'(z) \Rightarrow$$
$$\Rightarrow Q(y) = Q(z), Q'(y) = Q'(z) \Rightarrow$$
$$\Rightarrow \mathrm{Class}_Q(y) = \mathrm{Class}_Q(z), \mathrm{Class}_{Q'}(y) = \mathrm{Class}_{Q'}(z).$$

□

The operation common $\wedge$

The concept of coherence is required in order to introduce the operation common:

Definition 2 *Given a set Λ and a qualitative space $\mathbb{S}_n$, two qualitativizations of Λ, Q, Q' are* coherent, $Q \rightleftarrows Q'$, *iff*

$$Q(a_t) \cap Q'(a_t) \neq \emptyset, \quad \forall a_t \in \Lambda. \tag{2}$$

This last condition is equivalent to say that $Q(a_t) \approx Q'(a_t), \forall a_t \in \Lambda$.[1]

It is clear that the relation $\rightleftarrows$ is symmetric and reflexive.

Definition 3 *Given a set Λ and a qualitative space $\mathbb{S}_n$, the* set of coherent qualitativizations *of a qualitativization Q,* $\mathrm{Cohe}(Q)$, *is*

$$\mathrm{Cohe}(Q) = \{Q' \textit{qualitativization of } \Lambda \mid Q \rightleftarrows Q'\} \tag{3}$$

[1] In the theory of absolute orders of magnitude, two labels $\mathcal{E}, \mathcal{F}$ are qualitative equal, $\mathcal{E} \approx \mathcal{F}$, iff $\mathcal{E} \cap \mathcal{F} \neq \emptyset$.

Intuitively speaking, $\mathrm{Cohe}(Q)$ are all the qualitativizations having "some agreement" when they assign labels to all the elements of Λ.

Definition 4 *Given two qualitativizations Q and Q', such that $Q \rightleftarrows Q'$, the operation $Q \wedge Q'$ is a new qualitativization function $Q \wedge Q' : \Lambda \to \mathbb{S}_n$ such that*

$$(Q \wedge Q')(a_t) = Q(a_t) \cap Q'(a_t).$$

It is not difficult to check that the operations mix and common are commutative and associative, so it can be considered the mix and common operation of any number of qualitativizations $Q_1, \ldots, Q_n$.

An order relation can be defined from the operation common and mix:

Definition 5 *Given two qualitativizations Q and Q' of a set Λ over a qualitative space $\mathbb{S}_n$, Q* is less accurate *than Q', or $Q \leq Q'$, when $Q \vee Q' = Q$. That is to say that $\forall a_t \in \Lambda$ then $Q'(a_t) \subset Q(a_t)$, i.e. each element of the set Λ is more precise described by Q' than by Q.*

Entropy

The information of a label

The information of a label $\mathcal{E}$ will be a positive continuous real function on the measure of the label, and will be denoted by $I(\mathcal{E})$. It also will be assumed that if a label $\mathcal{E}$ is more precise than a label $\mathcal{E}'$, then there is more information in $\mathcal{E}$ than in $\mathcal{E}'$:

$$\mathcal{E} \leq_P \mathcal{E}' \Rightarrow I(\mathcal{E}) \geq I(\mathcal{E}').$$

Another assumption about the function I is that the information of the label **?** is zero.

The following definition of I inspired in the Shannon theory of information ((Shannon 1948)) verifies these assumptions:

Definition 6 *The information of a label $\mathcal{E} \in \mathbb{S}_n$ is*

$$I(\mathcal{E}) = \log \frac{1}{\mu(\mathcal{E})},$$

where μ is a normalized measure defined in $\mathbb{S}_n$ and $\mu(\mathcal{E}) \neq 0$.

It is trivial to check that it is positive and continuous, and decreases with respect to $\leq_P$:
From the definition of $\leq_P$ in expression (1) from the section 2:

$$\mathcal{E} \leq_P \mathcal{F} \Rightarrow \mathcal{E} \subset \mathcal{F} \Rightarrow \mu(\mathcal{E}) \leq \mu(\mathcal{F}) \Rightarrow \log \frac{1}{\mu(\mathcal{E})} \geq \log \frac{1}{\mu(\mathcal{F})}$$

Morover, $I(\textbf{?}) = \log 1 = 0$.
Example: In the classical $\mathbb{S}_n$ model, defining a measure $\mu([a_i, a_{i+1}]) = (a_{i+1} - a_i)/(a_n - a_1)$, the information of a label is $I([a_i, a_{i+1}]) = \log \left(\frac{a_n - a_1}{a_{i+1} - a_i} \right)$.

Entropy of a qualitativization in $\mathbb{S}_n$

Let us suppose a normalized measure $\overline{\mu}$ in the set Λ.

Definition 7 *The entropy H of a qualitativization Q is defined as:*

$$H(Q) = \sum_{\mathcal{E} \in \mathbb{S}_n} \overline{\mu}(Q^{-1}(\mathcal{E})) I(\mathcal{E}). \tag{4}$$

If $\Lambda/\sim_Q = \{X_i, i \in J\}$, that is, the set of equivalence classes of $\sim_Q$, then the expression 4 can be expressed as

$$H(Q) = \sum_{i \in J} \overline{\mu}(X_i) I(Q(X_i)). \tag{5}$$

The expression of entropy in the definition (7) defines the entropy as a weighted average of the information of the elements of the set Λ given by Q.

Proposition 2 *Given a set Λ and the space $\mathbb{S}_n$, each with its own measure, the maximum entropy, $H(\tilde{Q})$, is achieved when $Q(\Lambda) = \{\mathcal{E}_*\}$ where $\mathcal{E}_*$ is the shortest label with respect to μ. In other words, the maximum entropy is reached when Q maps the whole set Λ to the most precise label:*

$$H(\tilde{Q}) = \max_Q H(Q) = I(\min_{\mathcal{E}} \mu(\mathcal{E})) = \log \frac{1}{\mu(\mathcal{E}_*)}.$$

Proof: It is clear that the label with maximum information is the shortest label with respect to μ; if this label is called $\mathcal{E}_*$, then

$$H(Q) \leq \sum_{\mathcal{E} \in \mathbb{S}_n} \overline{\mu}(Q^{-1}(\mathcal{E})) I(\mathcal{E}_*) =$$
$$= I(\mathcal{E}_*) \sum_{\mathcal{E} \in \mathbb{S}_n} \overline{\mu}(Q^{-1}(\mathcal{E})) = I(\mathcal{E}_*),$$

because the measure $\overline{\mu}$ is normalized and the set $Q^{-1}(\mathcal{E})$ is a partition of Λ. □

According to this proposition it is possible to define the precision of a qualitativization:

Definition 8 *The* precision *of a qualitativization Q of a set Λ, $h(Q)$, is the relative entropy respect the maximum entropy $H(\tilde{Q})$ for the set Λ in $\mathbb{S}_n$*

$$h(Q) = \frac{H(Q)}{H(\tilde{Q})} \tag{6}$$

This quantity is a real number between 0 and 1, the closer it is to 1, the more accurate the evaluator is.

Lemma 1 *For all labels $\mathcal{E}, \mathcal{F} \in \mathbb{S}_n$ it is hold that $I(\mathcal{E} \sqcup \mathcal{F}) \leq I(\mathcal{E}) + I(\mathcal{F})$.*

Proof: Since $\mathcal{E} \leq_P \mathcal{E} \sqcup \mathcal{F}$ then $I(\mathcal{E}) \geq I(\mathcal{E} \sqcup \mathcal{F})$ and then $I(\mathcal{E} \sqcup \mathcal{F}) \leq I(\mathcal{E}) + I(\mathcal{F})$ □

From lemma 1 the next result with respect the operation mix of two qualitativizations is presented:

Theorem 1 *Given a set Λ, the space $\mathbb{S}_n$, and two qualitativizations Q and Q',then*

$$H(Q \vee Q') \leq H(Q) + H(Q').$$

Proof: From equation 4

$$H(Q \vee Q') = \sum_{\mathcal{F} \in (Q \vee Q')(\Lambda)} \overline{\mu}((Q \vee Q')^{-1}(\mathcal{F}))I(\mathcal{F}), \quad (7)$$

and using the proposition 1

$$(Q \vee Q')^{-1}(\mathcal{F}) = \bigcup_{X_i \in (\Lambda/\sim_Q) \cap (\Lambda/\sim_{Q'})} X_i.$$

Since this union is a disjoint union and $\overline{\mu}$ is a measure

$$\overline{\mu}((Q \vee Q')^{-1}(\mathcal{F})) = \sum_{i \in J} \overline{\mu}(X_i),$$

where J is an index set. Taking into account that $X_i \in (\Lambda/\sim_Q) \cap (\Lambda/\sim_{Q'})$, it can be expressed as $X_i = M_{j_i} \cap N_{k_i}$ where $M_{j_i} \in \Lambda/\sim_Q$ and $N_{k_i} \in \Lambda/\sim_{Q'}$. By construction of $\Lambda/_{Q \vee Q'}$, each label is $\mathcal{F} = Q(M_{j_i}) \sqcup Q'(N_{j_i})$, then

$$\overline{\mu}((Q \vee Q')^{-1}(\mathcal{F}))I(\mathcal{F}) =$$
$$= \sum_{i \in J} \overline{\mu}(M_{j_i} \cap N_{k_i}) I(Q(M_{j_i}) \sqcup Q'(N_{j_i})),$$

from the lemma 1:

$$\overline{\mu}((Q \vee Q')^{-1}(\mathcal{F}))I(\mathcal{F}) \leq$$
$$\leq \sum_{i \in J} \overline{\mu}(M_{j_i} \cap N_{k_i}) I(Q(M_{j_i}) + I(Q'(N_{j_i})),$$

Putting it all together into 7

$$H(Q \vee Q') \leq \sum_{M \in \Lambda/\sim_Q, N \in \Lambda/\sim_{Q'}} \overline{\mu}(M \cap N)(I(Q(M)) + I(Q(N))),$$

On the other hand $M \cap N \subset M, N$ so $\overline{\mu}(M \cap N) \leq \overline{\mu}(M), \overline{\mu}(N)$ whence the inequality is inferred. □

The next proposition shows that the entropy respects the accuracy relation between qualitativizations:

Proposition 3 *Given a set Λ, the space $\mathbb{S}_n$, and two qualitativizations Q and Q' such that $Q \leq Q'$ then $H(Q) \leq H(Q')$.*

Proof: Lets write $\Lambda/\sim_Q = \cup_{i \in M} X_i$, $\Lambda/\sim_Q = \cup_{j \in N} Y_j$, and $(\Lambda/\sim_Q) \cap (\Lambda/\sim_{Q'}) = \cup_{i,j}(X_i \cap Y_j)$. For each $X_i \in \Lambda/\sim_Q$ there exist a subset of index $N_i \subset N$ such that $X_i = \cup_{j \in N_i}(X_i \cap Y_j)$ and vice-versa, there exist a subset of index $M_j \subset M$ such that $Y_j = \cup_{i \in M_j}(X_i \cap Y_j)$ (all unions are disjoint unions). If $X_i \cap Y_j \neq \emptyset$ then from definition 5:

$$Q'(Y_j) \subset Q(X_i) \Rightarrow I(Q(X_i)) \leq I(Q'(Y_j)) \quad (8)$$

The entropy of Q is

$$H(Q) = \sum_{i \in M} \overline{\mu}(X_i) I(Q(X_i)) =$$
$$= \sum_{i \in M} \overline{\mu}\left(\cup_{j \in N_i}(X_i \cap Y_j)\right) I(Q(X_i)) =$$
$$= \sum_{i \in M} \left(\sum_{j \in N_i} \overline{\mu}(X_i \cap Y_j) I(Q(X_i)) \right) \leq$$

from the inequality in (8)

$$\leq \sum_{i \in M} \left(\sum_{j \in N_i} \overline{\mu}(X_i \cap Y_j) I(Q(Y_j)) \right) =$$
$$= \sum_{j \in N_i} \left(\sum_{i \in M} \overline{\mu}(X_i \cap Y_j) I(Q(X_i)) \right) =$$
$$= \sum_{j \in N} \overline{\mu}(Y_j) I(Q(Yj)) = H(Q').$$

□

Coherence degree in group decision

The measure of the precision and coherence in group decision evaluation problems is one of the main applications of the theory presented in this paper. The underlying idea on the next definition stands on the need to measure the precision of a set of evaluators and the coherence degree of its evaluations when they are evaluating a set by means of labels belonging to a $\mathbb{S}_n$.

First of all there is a formalization of the problem of the group evaluation of a set: Given a space $\mathbb{S}_n$, a finite non empty set $\Lambda = \{a_1, \ldots, a_N\}$ and set $\mathbb{E} = \{\alpha_1, \ldots, \alpha_M\}$, (it is the set of group evaluators), a *group evaluation* of Λ is the pair $(\Lambda, \mathcal{Q}_{\mathbb{E}})$, where $\mathcal{Q}_{\mathbb{E}} = \{Q_i : \Lambda \to \mathbb{S}_n \mid i \in \mathbb{E}\}$.

There exists *coherence* in the group, if and only if, the group is coherent, i.e. iff $\forall Q \in \mathcal{Q}_{\mathbb{E}}, \mathrm{Cohe}(Q) = \mathcal{Q}_{\mathbb{E}}$. Notice that it is evident that the last condition is satisfied if there exists a Q such that $\mathrm{Cohe}(Q) = \mathcal{Q}_{\mathbb{E}}$. Assuming that the group is in coherence, the next definition of coherence degree measures the relation between the entropy of operations mix and common in the qualitativizations of the group:

Definition 9 *Given a group evaluation $(\Lambda, \mathcal{Q}_{\mathbb{E}})$ in coherence, the* coherence degree *of the group, $\kappa(\mathcal{Q}_{\mathbb{E}})$, is*

$$\kappa(\mathcal{Q}_{\mathbb{E}}) = \frac{H(\bigvee_{i \in \mathbb{E}} Q_i)}{H(\bigwedge_{i \in \mathbb{E}} Q_i)} \quad (9)$$

When the whole group qualitativizes the set Λ in the same way, i.e., when $Q_i = Q_j, \forall i, j \in \mathbb{E}$, then $\kappa(\mathcal{Q}_{\mathbb{E}}) = 1$, and if $\kappa(\mathcal{Q}_{\mathbb{E}}) = 1$ then $Q_i = Q_j, \forall i, j \in \mathbb{E}$. On the other hand, the spread with Q_i, implies a small $H(\bigvee_{i \in \mathbb{E}} Q_i)$ and a big $H(\bigwedge_{i \in \mathbb{E}} Q_i)$. The given degree of coherence will give us a global index with respect to the whole group of evaluators. The key point on this definition is that the closer this degree is to 1, the closer the group is to be in a consensus relation.. When the coherence degree is not satisfactory, an iterative process will start to increase this degree.

The next property shows that the coherence degree of a group evaluation problem cannot increase by adding to the group a new evaluator.

Proposition 4 *Consider a group evaluation* $(\Lambda, \mathcal{Q}_{\mathbb{E}})$ *in coherence. Let be* Q_{new} *a new evaluator of* Λ *such that* $Q_{\text{new}} \notin \mathcal{Q}_{\mathbb{E}}$*, then*

$$\kappa(\mathcal{Q}_{\mathbb{E}} \cup \{Q_{\text{new}}\}) \leq \kappa(\mathcal{Q}_{\mathbb{E}}).$$

Proof:: From the definitions 4 and 5 can be deduced the inequalities $Q \vee Q' \leq Q, Q' \leq Q \wedge Q'$ whence can be deduced that if a new evaluator joints the group of evaluators then:

$$(\vee_{i\in\mathbb{E}} Q_i) \vee Q_{\text{new}} \leq \vee_{i\in\mathbb{E}} Q_i,$$
$$\wedge_{i\in\mathbb{E}} Q_i \leq (\wedge_{i\in\mathbb{E}} Q_i) \wedge Q_{\text{new}}.$$

From proposition

$$H(\vee_{i\in\mathbb{E}} Q_i) \vee Q_{\text{new}}) \leq H(\vee_{i\in\mathbb{E}} Q_i),$$
$$H(\wedge_{i\in\mathbb{E}} Q_i) \leq H((\wedge_{i\in\mathbb{E}} Q_i) \wedge Q_{\text{new}}),$$

whence $\kappa(\mathcal{Q}_{\mathbb{E}} \cup \{Q_{\text{new}}\}) \leq \kappa(\mathcal{Q}_{\mathbb{E}})$. □

Therefore, the only way to increase the coherence degree in a group is that the evaluators in the group reconsider the problem.

Conclusions and future research

A mathematical framework is presented to define group decision techniques to measure precision and coherence based on a qualitative structure of orders of magnitude.

This paper introduces the concept of entropy by means of absolute orders of magnitude qualitative spaces to measure the amount of information of a system when using orders of magnitude descriptions to represent it. On the other hand, entropy makes it possible to introduce a measure of coherence in group decision-making problems.

The obtained results can be applied to tackle evaluation and ranking problems which require an ordinal set of labels to qualify decision alternatives.

A coherence degree is introduced in order to obtain an objective measure of reliability in group decision making to detect incoherencies and avoid potential subjectivity caused by conflicts of interest regarding evaluators.

From a theoretical point of view, future research could focus on two lines. On the one hand, it could focus on the analysis of the given structure of the qualitative descriptions of a system to define a lattice using mix and common operations. On the other hand a distance between qualitative descriptions will be defined by means of conditioned entropy.

Within the framework of applications, this work and its related methodology will be orientated towards the development of techniques to detect malfunctioning within an evaluation committee, and to analyse whether it can reflect a corruption or a lack of knowledge in a part of the committee.

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